Polygon Dynamics Under Reflection Operations

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Abstract

We study the dynamics of a polygon as its vertices are iteratively reflected across the perpendicular bisector of their neighbors. For inscribed polygons and uninscribed quadrilaterals, we prove conditions for congruence with the initial polygon, and disprove the result for general polygons.

1 Introduction

Consider any set of n points $\mathcal{P} = \{z_1, \ldots, z_n\}$ for $z_i \in \mathbb{R}^2$, and the corresponding n-polygon consisting of sides $z_i z_{i+1}$ for $i \in \{1, \ldots n\}$ where $z_{n+1} = z_1$ and $z_0 = z_n$. Then consider the dynamical system defined by operators $\{r_1, \ldots r_n\}$ where $r_i(\mathcal{P})$ reflects vertex z_i across the perpendicular bisector of its two neighbors z_{i-1} and z_{i+1} to obtain z_i', which replaces z_i in $\mathcal{P}$. We refer to r_i as a single *reflection operation*, described below in Figure 1, and write $\mathcal{P}^k = \{z_1^k, \ldots, z_n^k\}$ to denote $\mathcal{P}$ after k reflection operations.

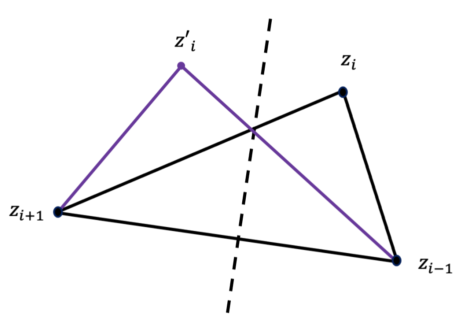

Figure 1: A reflection operation applied to point $z_i \in \mathcal{P}$ to obtain z_i', where the dashed line is the perpendicular bisector of z_{i+1} and z_{i-1}.

As demonstrated in Figure 1, the reflection operation does not change a vertex's adjacencies, but essentially swaps the side lengths connecting z_{i-1} to z_i, and z_i to z_{i+1}, respectively. To formalize this notion, first define the n−tuple

$$\mathcal{S}^k = (S_1^k, \dots, S_{n-1}^k, S_n^k) = (|z_2^k - z_1^k|, \dots, |z_n^k - z_{n-1}^k|, |z_1^k - z_n^k|)$$

that contains the side lengths of the polygon defined by $\mathcal{P}^k$. If r_i is applied at iteration $k+1$,

$$\mathcal{S}^{k+1} = \{S_1^k, \dots, S_{i+1}^k, S_i^k, \dots, S_n^k\}.$$

Unsurprisingly, because perimeter is the sum of the individual side lengths the perimeter of the polygon is invariant under r_i. However, other properties such as convexity and self-intersection can be violated for a general n points. For example, Figure 2 and Figure 3, show polygons that become self intersecting and concave, respectively, after a reflection operation.

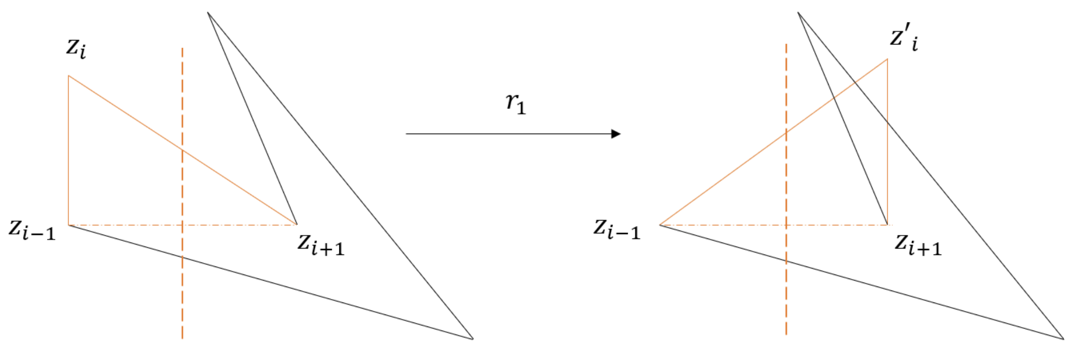

Figure 2: A reflection resulting in a self-intersecting polygon.

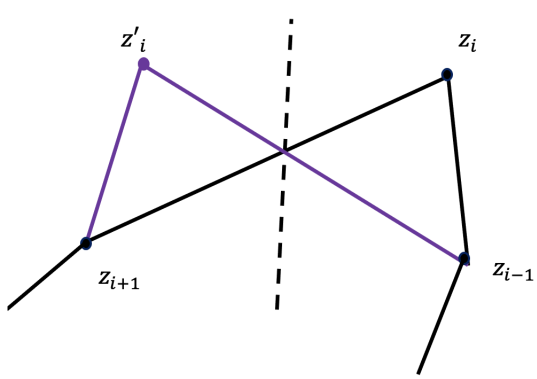

Figure 3: A reflection resulting in a convex polygon becoming concave.

To study what properties of $\mathcal{P}^0$ and $\mathcal{P}^k$ produce congruent polygons, where two sets or polygons are congruent if one can be transformed into the other by an isometry – a sequence of rigid motions – we first state conditions on the initial set $\mathcal{P}^0$. Specifically, we assume that $\mathcal{P}^0$ is inscribed on a circle and that the initial points are arranged in increasing order from $[0, 2\pi]$ along its circumference, described in detail in Section 2. These initial conditions will subsequently open the door to proving conditions on $\mathcal{P}$ that produce congruence between the initial and final polygon after some number $k \geq 0$ of reflection operations. In particular,

Theorem 1. *Given an inscribed set $\mathcal{P}^k$ for $k \geq 0$ with ordered initial points given in Equation 1, if $S^k = S^0$, then the polygons defined by $\mathcal{P}^k$ and $\mathcal{P}^0$ are congruent.*

We then relax these initial conditions to find for a non inscribed quadrilateral,

Theorem 2. *Given a $\mathcal{P}^0$ such that $|\mathcal{P}^0| = 4$, if $\mathcal{S}^0 = \mathcal{S}^k$ then the polygons defined by $\mathcal{P}_0$ and $\mathcal{P}_k$ are congruent.*

Both proofs will rely on bridging the gap between the dynamical system to the geometry, then the geometry to the algebra. For the inscribed case, we first formalize the initial conditions on $\mathcal{P}^0$ in Section 2. We then describe invariants of $\mathcal{P}$ in Section 3, namely inscribeability, area, and convexity. After proving under what conditions we can conclude that the final polygon is congruent to the initial polygon in Section 4, we find an algebraic representation of the reflection operation in Section 5. In Section 6, we remove these initial conditions in the special case when $|\mathcal{P}| = 4$ to prove a condition on congruence, then show a counter example for this proof in the general case of $n \neq 4$ in Section 7.

2 Initial conditions on $\mathcal{P}^0$

We first require that $\mathcal{P}^0$ is inscribed on a circle. Recall the general case of $\mathcal{P} = \{z_1, \dots, z_n\}$ for $z_i \in \mathbb{R}^2$, and consider the polar representation $z_i = s_i e^{i\theta_i}$ for radius s_i and angle $\theta_i \in [0, 2\pi]$. As $\mathcal{P}^0$ is inscribed, we can then assume without loss of generality for $\mathcal{P}^0$ that $s_i = 1$ for all $i = 1, \dots, n$. Therefore, we can fully characterize z_i^0 in terms of θ_i^0 measured with respect to an arbitrary point on the unit circle. In doing so, we essentially reduce the dimension from $\mathcal{P}^0 \in \mathbb{R}^{2n}$ to $\mathcal{P}^0 \in [0, 2\pi]^n$, and we proceed to write $\widetilde{\mathcal{P}^0} = \{\theta_1^0, \dots, \theta_n^0\}$ and use the same set of operators $\{r_1, \dots, r_n\}$. Using this assumption, we'll later show that $\mathcal{P}^k$ is also inscribed for all $k \geq 0$.

Our second assumption is that the θ_i^0 are arranged in increasing order from $[0, 2\pi]$ along the circumference of the unit circle,

$$0 \le \theta_1^0, \le \ldots \le \theta_n^0 \le 2\pi, \tag{1}$$

where we say that $\widetilde{\mathcal{P}^0}$ is *ordered* if Equation 1 holds.

Notice that this change of variables preserves the properties of the original representation. In particular, $\widetilde{\mathcal{P}^0}$ still defines a polygon when we form line segments between points $(1, \theta_i)$ and $(1, \theta_{i+1})$, and these segments will now form chords on the unit circle subtended by a central angle of $|\theta_{i+1} - \theta_i|$. In general, since the length of a chord on the unit circle with central angle of θ is given as $2\sin\left(\frac{\theta}{2}\right)$, we can completely characterize the side lengths of the polygon defined by $\widetilde{\mathcal{P}^0}$ similarly as before using the n-tuple

$$\widetilde{\mathcal{S}} = (\widetilde{S}_1, \ldots, \widetilde{S}_{n-1}, \widetilde{S}_n) = (|\theta_2 - \theta_1|, \ldots, |\theta_n - \theta_{n-1}|, |\theta_1 - \theta_n|).$$

3 Invariants of an inscribed polygon

We now proceed under the assumption that $\widetilde{\mathcal{P}}^0$ is inscribed on a unit circle, and write $\widetilde{\mathcal{P}}^0 = \{\theta_1^0, \ldots, \theta_n^0\}$ for $0 \le \theta_1^0, \le \ldots \le \theta_n^0 \le 2\pi$. However, in order to leverage these assumptions in our proof of the polygon's invariants, we first must prove that these assumptions are invariant themselves under reflection operations.

3.1 Invariance of assumptions

Lemma 1. *Given $\widetilde{\mathcal{P}}^0$ inscribed on a circle with $\widetilde{\mathcal{P}}^0 = \{\theta_1^0, \ldots, \theta_n^0\}$, then $\mathcal{P}^k$ for $k \ge 0$ is also inscribed on a circle, and the center is invariant.*

Proof. We proceed using induction on k, where the base case holds by assumption. Now consider the set $\mathcal{P}^k = \{z_1^k, \ldots, z_n^k\}$ after k reflection operations, and recall that $r_i(\mathcal{P}^k)$ reflects z_i^k over the perpendicular bisector of z_{i-1}^k and z_{i-1}^k.

Denoting the center of the circle by point O, consider the segment $\overline{Oz_i^k}$. Because a reflection over a line in Euclidean space is distance preserving by definition of reflections being an isometry, if we obtain the points O' and z_i^{k+1} after a reflection we must have that $|\overline{Oz_i^k}| = |\overline{O'z_i^{k+1}}|$. Because the perpendicular bisector of the chord $z_{i-1}^k z_{i-1}^k$ goes through O, then $O = O'$. Therefore $|\overline{Oz_i^k}| = |\overline{Oz_i^{z_i^{k+1}}}|$. Therefore it follows that the segment $\overline{Oz_i^{k+1}}$ is also a radius on the circle, and so $\mathcal{P}^{k+1} = \{z_1^{k+1}, \ldots, z_n^{k+1}\}$ is also inscribeable and can therefore be written as $\mathcal{P}^{k+1} = \{\theta_1^{k+1}, \ldots, \theta_n^{k+1}\}$. This is shown in Figure 4 below. □

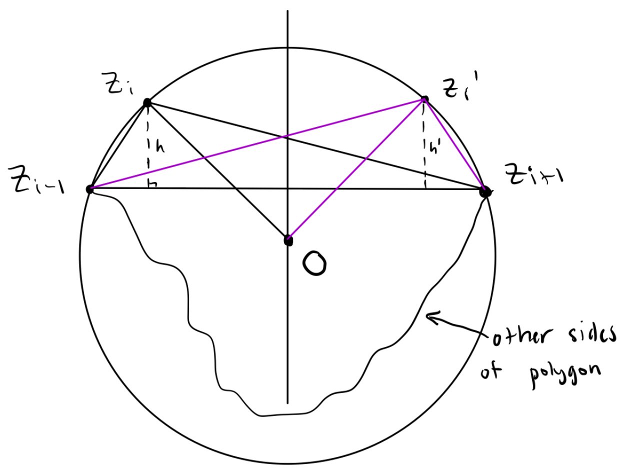

Figure 4: Reflecting z_i to z_i' in a reflection operation.

In essence, what we have shown is that our first assumption on the inscribeability of $\widetilde{\mathcal{P}}^0$ is invariant. Therefore, we can freely use the notation that $\widetilde{\mathcal{P}}^k = \{\theta_1^k, \ldots, \theta_n^k\}$ for all $k \geq 0$. We now proceed to do the same for our second assumption on the ordering of the initial angles θ_i^0.

Theorem 3. *Given $\widetilde{\mathcal{P}}^0 = \{\theta_1^0, \ldots, \theta_n^0\}$ inscribed on a circle, if $\widetilde{\mathcal{P}}^0$ is ordered $0 \leq \theta_1^0, \leq \ldots \leq \theta_n^0 \leq 2\pi$, then for every $k \geq 0$ there exists a rotation R_α by α degrees such that $0 \leq R_\alpha(\theta_1^k), \leq \ldots \leq R_\alpha(\theta_n^k) \leq 2\pi$.*

Since we are essentially considering $\theta_1, \ldots, \theta_n$ modulo 2π, it is not hard to see that it is possible that at some iteration we reflect θ_n over $\theta = 0 = 2\pi$ to obtain $0 \leq \theta_n \leq \theta_1$, and hence the need for the rotation R_α.

Proof. We again proceed by induction, where the base case holds by assumption. Consider $\widetilde{\mathcal{P}}^k = \{\theta_1^k, \ldots, \theta_n^k\}$ which we assume to be ordered $\theta_1^k \leq \ldots \leq \theta_n^k$. Setting $\alpha = \theta_{i-1}^k$ and applying the rotation function R_α to all n points,

$$0 = R_\alpha(\theta_{i-1}^k) \leq R_\alpha(\theta_i^k) \leq R_\alpha(\theta_{i+1}^k).$$

Now apply the reflection operation to these rotated points to obtain $R_\alpha(\theta_i^k)'$. Noting that the perpendicular bisector of $R_\alpha(\theta_{i-1}^k)$ and $R_\alpha(\theta_{i+1}^k)$ is also the perpendicular bisector of $R_\alpha(\theta_i^k)$ and $R_\alpha(\theta_i^k)'$ the midpoints of these two arcs are also the same, therefore

$$\frac{R_\alpha(\theta_{i+1}^k) + R_\alpha(\theta_{i-1}^k)}{2} = \frac{R_\alpha(\theta_i^k) + R_\alpha(\theta_i^k)'}{2}$$

so

$$R_\alpha(\theta_i^k)' = R_\alpha(\theta_{i+1}^k) + R_\alpha(\theta_{i-1}^k) - R_\alpha(\theta_i^k). \tag{2}$$

Using $R_\alpha(\theta_{i-1}^k) = 0$ and $R_\alpha(\theta_{i+1}^k) \geq R_\alpha(\theta_i^k)$, it follows that

$$R_\alpha(\theta_{i-1}^k) \leq R_\alpha(\theta_i^k)' \leq R_\alpha(\theta_{i+1}^k).$$

As the rest of the n points aside from θ_i^k are unaffected by a reflection operation to θ_i^k, it follows that there exists a rotation R_α by α degrees such that

$$0 \leq R_\alpha(\theta_1^k), \leq \ldots \leq R_\alpha(\theta_n^k) \leq 2\pi.$$

□

Similar to inscribeability, the relative ordering of points is invariant under reflection operations. Aside from helping us prove invariants, this result will also facilitate a proof of congruence in Section 4 and a closed form matrix representation of r_i in Section 5.

3.2 Invariance of polygon properties

Until now we have not bridged the gap between the initial assumptions on $\mathcal{P}^0$ and the resulting polygon. However, we now do so by turning our attention to invariants of the polygon under reflection operations.

Lemma 2. *Given an inscribed and ordered set $\widetilde{\mathcal{P}}^k = \{\theta_1^k, \ldots, \theta_n^k\}$ for $k \geq 0$, let A^k denote the area of the polygon defined by $\widetilde{\mathcal{P}}^k$, then A^k is invariant under reflection operations.*

Proof. By Theorem 3, there exists a rotation R_α such that

$$0 \leq R_\alpha(\theta_1^k), \leq \ldots \leq R_\alpha(\theta_n^k) \leq 2\pi.$$

As this rotation is area preserving, we proceed to assume without loss of generality that $0 \leq \theta_1^k \leq \ldots \leq \theta_n^k \leq 2\pi$.

We prove that A^k is invariant by proving the area contained within the circle but not contained within the polygon is constant, formally that the total area of the *segments* is constant, where a single segment is shown in Figure 5.

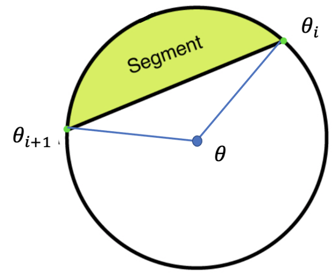

Figure 5: A single segment defined by the chord between θ^k_{i+1} and θ^k_i

Recall that the area of a segment with interior angle θ is given as $\frac{1}{2}(\theta - \sin\theta)$ and let A^k_i denote the area of the segment formed with interior angle $\theta^k_{i+1} - \theta^k_i$. Therefore $A^k = \sum_{i=1}^n A_i$. Since it holds at each reflection operation that

$$\widetilde{\mathcal{S}}^{k+1} = \{\widetilde{S}^k_1, \dots, \widetilde{S}^k_{i+1}, \widetilde{S}^k_i, \dots, \widetilde{S}^k_n\}$$

where $\widetilde{S}_i = |\theta_{i+1} - \theta_i|$, it follows that $A^k_{i-1} = A^{k+1}_i$ and $A^k_i = A^{k+1}_{i-1}$, thus A^k is invariant. □

Claim 1. *Given an inscribed and ordered set $\widetilde{\mathcal{P}}^k = \{\theta^k_1, \dots, \theta^k_n\}$ for $k \geq 0$, then the polygon defined by $\widetilde{\mathcal{P}}^k$ is convex for all $k \geq 0$.*

We do not provide a full proof of this claim here, but one can do so by showing the intersection of n convex sets, each itself the intersection between a half-space formed using points θ_i and θ_{i+1} and the unit circle, is also convex.

4 Congruence for inscribed and ordered $\widetilde{\mathcal{P}}^0$

We now turn our attention to under which conditions on $\widetilde{\mathcal{P}}^k$ can we conclude that the two polygons associated with $\widetilde{\mathcal{P}}^0$ and $\widetilde{\mathcal{P}}^k$ are congruent. As opposed to relying on geometric properties of the two polygons, this proof also uses a rotation function R_α similar to the proof of the invariant of the ordering assumption in Theorem 3. First, recall that we define the n−tuple

$$\widetilde{\mathcal{S}}^k = (\widetilde{S}^k_1, \dots, \widetilde{S}^k_{n-1}, \widetilde{S}^k_n) = (|\theta^k_2 - \theta^k_1|, \dots, |\theta^k_n - \theta^k_{n-1}|, |\theta^k_1 - \theta^k_n|).$$

to be the side lengths of the polygon defined by $\mathcal{P}^k$.

Theorem 4. *Given an inscribed and ordered set $\widetilde{\mathcal{P}}^k = \{\theta^k_1, \dots, \theta^k_n\}$ for $k \geq 0$, then if $\widetilde{\mathcal{S}}^k = \widetilde{\mathcal{S}}^0$ then the two polygons defined by $\widetilde{\mathcal{P}}^k$ and $\widetilde{\mathcal{P}}^0$ are congruent.*

Proof. Recall from Section 1 that we define two sets $\widetilde{\mathcal{P}}^k$ and $\widetilde{\mathcal{P}}^0$, and thus their corresponding polygons, to be congruent if their exists a distance preserving isometry between them. We show this is the case using Theorem 3 to show that there exists a rotation R_α such that $R_\alpha(\widetilde{\mathcal{P}}^k) = \widetilde{\mathcal{P}}^0$.

Letting $\alpha = \theta_1^k$, we obtain

$$R_\alpha(\widetilde{\mathcal{P}}^k) = \{0, \theta_2^k - \theta_1^k, \ldots, \theta_n^k - \theta_1^k\},$$

and from Theorem 3 it then follows that

$$0 \leq \theta_2^k - \theta_1^k \leq \ldots, \leq \theta_n^k - \theta_1^k \leq 2\pi. \tag{3}$$

We now claim that

$$R_\alpha(\widetilde{\mathcal{P}}^k) = \{0, \theta_2^k - \theta_1^k, \ldots, \theta_n^k - \theta_1^k\} = \widetilde{\mathcal{P}}^0$$

where it suffices to prove that for a general $i \in 1 \ldots, n$ that $\theta_i^k - \theta_1^k = \theta_i^0$. Observing that

$$\theta_i^k - \theta_1^k = (\theta_i^k - \theta_{i-1}^k) + (\theta_{i-1}^k - \theta_{i-2}) \ldots + (\theta_2^k - \theta_1^k) \tag{4}$$

we can use Equation 3 to note for all $j \in \{2, \ldots, n\}$ that

$$(\theta_j^k - \theta_{j-1}^k) \geq 0$$

so

$$(\theta_j^k - \theta_{j-1}^k) = |\theta_j^k - \theta_{j-1}^k| = \widetilde{S}_{j-1}^k.$$

Therefore, we can rewrite Equation 4 as

$$\theta_i^k - \theta_1^k = \widetilde{S}_{i-1}^k + \ldots + \widetilde{S}_1^k$$

By assumption of $\widetilde{\mathcal{S}}^k = \widetilde{\mathcal{S}}^0$, we have that $\widetilde{S}_j^k = \widetilde{S}_j^0$ for all j, so

$$\widetilde{S}_{i-1}^k + \ldots + \widetilde{S}_1^k = \widetilde{S}_{i-1}^0 + \ldots + \widetilde{S}_1^0$$

and by assumption of the ordering of $\mathcal{P}^0$, we similarly have that

$$\widetilde{S}_{i-1}^0 + \ldots + \widetilde{S}_1^0 = \theta_i^0 - \theta_1^0$$

Applying a rotation to $\widetilde{\mathcal{P}}^0$ by θ_1^0 degrees gives that $R_\beta(\theta_1^0) = 0$, thus we can assume without loss of generality that $\theta_1^0 = 0$. We have thus shown that $\theta_i^k - \theta_1^k = \theta_i^0$, and thus there exists an isometry between $\widetilde{\mathcal{P}}^0$ and $\widetilde{\mathcal{P}}^k$, producing congruence between the two sets and their associated polygons. □

5 Algebraic representation

In addition to the invariants proven above, reflection operations performed on inscribed sets are can be described by closed form matrix expressions.

5.1 Representation using reflection matrices

Since $\mathcal{P}^0$ being initially inscribed implies that $\mathcal{P}^k$ is also inscribed on a circle, we can think of the reflection operators $\{r_1, \ldots, r_n\}$ as simply rotating vertices of the polygon around the unit circle at each step

We first noted in the proof of the invariancy of the ordering assumption in 3 that considering $\mathcal{P}$ modulo 2π presents difficulty when seeking to characterize θ_i^{k+1} in terms of $\theta_{i-1}^k, \theta_i^k, \theta_{i+1}^k$. However, we addressed this issue by using a rotation function R_α to essentially orient the location on unit circle where $0 = 2\pi$ in a way that avoid these issues.

In fact, recall from Theorem 3 the intermediate step that

$$R_\alpha(\theta_i^k)' = R_\alpha(\theta_{i+1}^k) + R_\alpha(\theta_{i-1}^k) - R_\alpha(\theta_i^k).$$

If we apply the reverse rotation R_α^{-1} then we recover the original orientation of the points, less θ_i, with respect to the unit circle.

However, it would be incorrect to assume that we can take such an inverse while still operating modulo 2π. Instead, if we revise our initial ordering condition to be

$$-\infty < \theta_1^0 \leq \ldots \leq \theta_n^0 < \infty$$

we avoid the modulo 2π issue and can take the inverse of Equation 2 to obtain

$$\theta_i^{k+1} = \theta_{i+1}^k + \theta_{i-1}^k - \theta_i^k.$$

Next, if we instead consider $\mathcal{P}$ to be a vector as opposed to a set, so $\mathcal{P} = [\theta_1^k \ \ldots \ \theta_n^k]^\intercal$, then the reflection operators $\{r_1, \ldots, r_n\}$ we've been using now behave as a family of n *reflection matrices* written $\{T_1, \ldots, T_n\}$.

We construct the i^{th} reflection matrix row by row as follows. Each row j other than the i^{th} row will simply be the j^{th} unit vector. For row i, the i^{th} element will be a -1, while the $(i+1)^{th}$ and $(i-1)^{th}$ element will be 1, of course when considered cyclically. We present a full implementation of this algorithm in the appendix. Consider the case of $n = 4$ below.

$$T_1 = \begin{bmatrix} -1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$T_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & -1 \end{bmatrix}.$$

5.2 Properties of reflection matrices

Observation 1. $T_i = T_i^{-1}$.

Proof. Note that the the perpendicular bisector of θ_{i-1} and θ_{i+1} is unchanged after a reflection operation on θ_i. Therefore, applying a second reflection operation reflects θ_i' across the same line, returning to θ_i, and thus therefore $T_iT_i = \mathbb{I}$ from which it follows that $T_i = T_i^{-1}$ since T_i has an inverse. □

Observation 2. *For* $|i - j| \mod (n-1) \geq 2$, $T_iT_j = T_jT_i$ *and* $(T_iT_j)^2 = \mathbb{I}$.

We require $|i - j| \mod (n-1) \geq 2$ so that the points $(1, \theta_i)$ and $(1, \theta_{i+1})$ are not adjacent.

Proof. It holds that $\theta_j \neq \theta_{i-1}$ and $\theta_j \neq \theta_{i+1}$. Therefore applying a reflection operation on θ_i does not change the perpendicular bisector of θ_j thus, the order that we apply the reflection operations is arbitrary, and so $T_iT_j = T_jT_i$. Considering the sequence of reflection operations $T_iT_jT_iT_j$, it follows that $T_iT_jT_iT_j = T_iT_iT_jT_j$. Using that $T_i = T_i^{-1}$ gives $T_iT_iT_jT_j = \mathbb{I}$. □

6 A special case: Quadrilateral congruence

Previously, while analyzing congruence relations on n-polygons, we simplified the problem such that the polygons were inscribed in a circle and non-self-intersecting. In this section, we will lift these restrictions and focus on polygons with four sides and show the following result.

Theorem 5. *Given a* $\mathcal{P}^0$ *such that* $|\mathcal{P}^0| = 4$, *if* $\mathcal{S}^0 = \mathcal{S}^k$ *then the polygons defined by* $\mathcal{P}^0$ *and* $\mathcal{P}^k$ *are congruent.*

Proof. First, note that the sum of opposite angles in a quadrilateral remains invariant under any of the operations r_1, r_2, r_3, r_4. Since angle α doesn't change, the sum of opposite angles remains $\alpha + \gamma = C$, where C is some constant. Figure 6 illustrates this. Similarly, the sum of the other two opposite angles remains constant, equal to $2\pi - \alpha - \gamma = 2\pi - C$. Note that there are two pairs of opposite angles, the sum of one pair is equal or less π, and the sum of the other pair will be equal or greater than π. Without loss of generality, assume that $\alpha + \gamma = C \le \pi$.

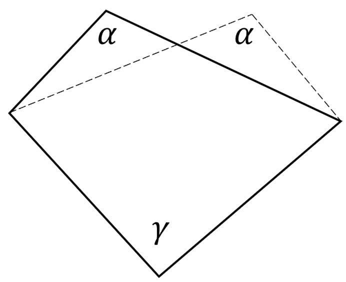

Figure 6: Sum of opposite angles is invariant under an operation.

Now, consider the polygon $\mathcal{P}^k$ we get after a sequence of operations (Figure 7). Recall that the sum of opposite angles remains constant; therefore, $\alpha + \gamma = \alpha' + \gamma' = C$, then $\gamma' = C - \alpha'$.

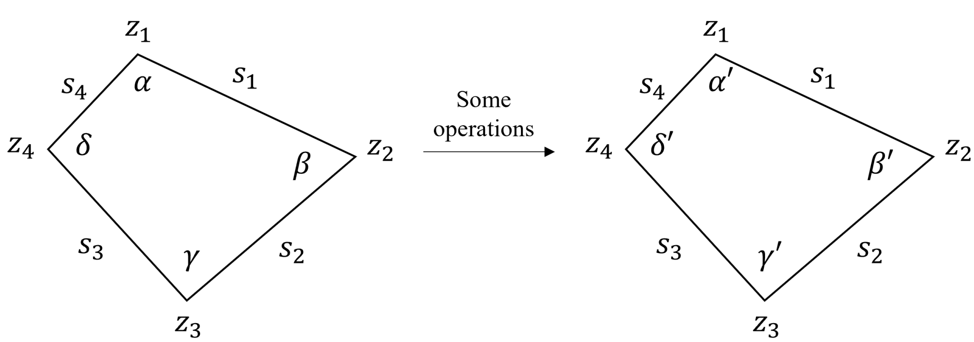

Figure 7: Getting back to the same permutation of the sides after some operations.

Now we will show that there is a unique possible value for angle α'. Consider the triangles $z_1z_2z_4$ and $z_2z_3z_4$. We know they have to be such that $\overline{z_2z_4}$ has the same length in both triangles. Figure 8 illustrates this.

Consider the triangles $z_1z_2z_4$ and $z_2z_3z_4$ as shown in Figure 8. In in triangle $z_1z_2z_4$, let the length of $\overline{z_2z_4}$ be a function of α'. As α' goes from 0 to π, the length of z_2z_4 increases. Similarly, in triangle $z_2z_3z_4$, let the length of z_2z_4 be a function of α'. As α' increases, $C - \alpha'$ decreases, and the length of z_2z_4 decreases.

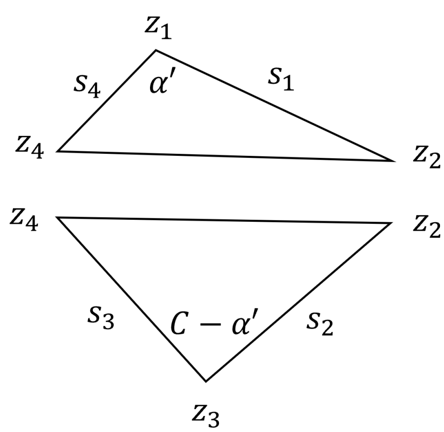

Figure 8: $\overline{z_2z_4}$ must have the same length in both triangles.

Since one function is strictly increasing and the other strictly decreasing, they only intersect at one point, as shown in Figure 9. Therefore, α' has only one possible value. But we already know there is a value that satisfies this system (this quadrilateral), and that value is α. Therefore $\alpha' = \alpha$, and $\gamma' = C - \alpha' = C - \alpha = \gamma$.

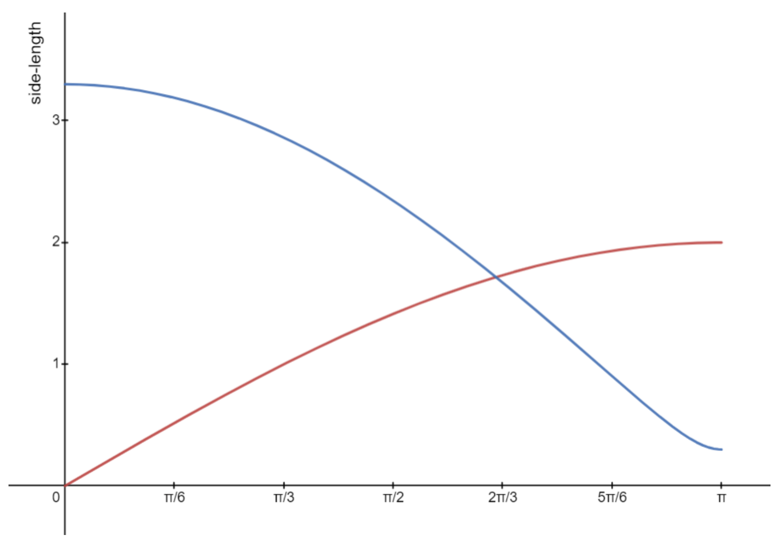

Figure 9: Quadrilateral diagonal as a function of α'.

Now that we have these uniquely defined triangles, we only need to glue them by the common diagonal z_2z_4 to get back the polygon they describe. Note this gluing operation can only be done in one way since we need to preserve both the order of sides (s_1, s_2, s_3, s_4) and the sum of opposite angles. $\square$

7 Counter example for the general case

Previously, we have shown in Theorem that 4 given an inscribed and ordered set $\widetilde{\mathcal{P}}^k = \{\theta_1^k, \ldots, \theta_n^k\}$ for $k \geq 0$, then if $\widetilde{\mathcal{S}}^k = \widetilde{\mathcal{S}}^0$ then the two polygons defined by $\widetilde{\mathcal{P}}^k$ and $\widetilde{\mathcal{P}}^0$ are congruent. We now provide a counter example in the general case

Consider the self-intersecting five-sided polygon in Figure 10, where $\mathcal{P} = \{z_1, .., z_5\}$ and the lengths of sides S_i^0 are labeled as i.

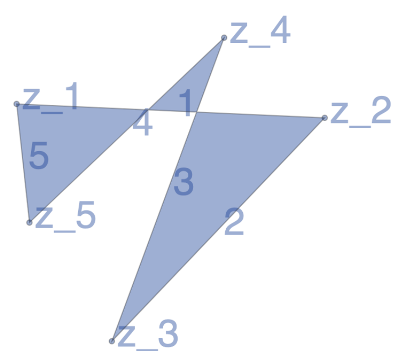

Figure 10: Initial polygon for counter-example

We will now apply a series of operators on $\mathcal{P}^0$, precisely $r_2r_3r_4r_5r_4r_3r_2r_1(\mathcal{P}^0)$ as shown in Figure 11. These operations produce $\mathcal{P}^8$, shown in the top right of Figure 11. At each step, the blue polygon represents $\mathcal{P}^k$, and the orange polygon represents the original polygon. Notice that in $\mathcal{P}^8$, all side lengths i are in between the points z_i and z_{i+1}, so we know $\mathcal{S}^8 = \mathcal{S}^0$.

However, despite the condition $\mathcal{S}^8 = \mathcal{S}^0$ being satisfied, clearly, we can see that the blue and orange polygons are not congruent, so Theorem 4 no longer holds in the general case when the assumptions in Section 2 are violated.

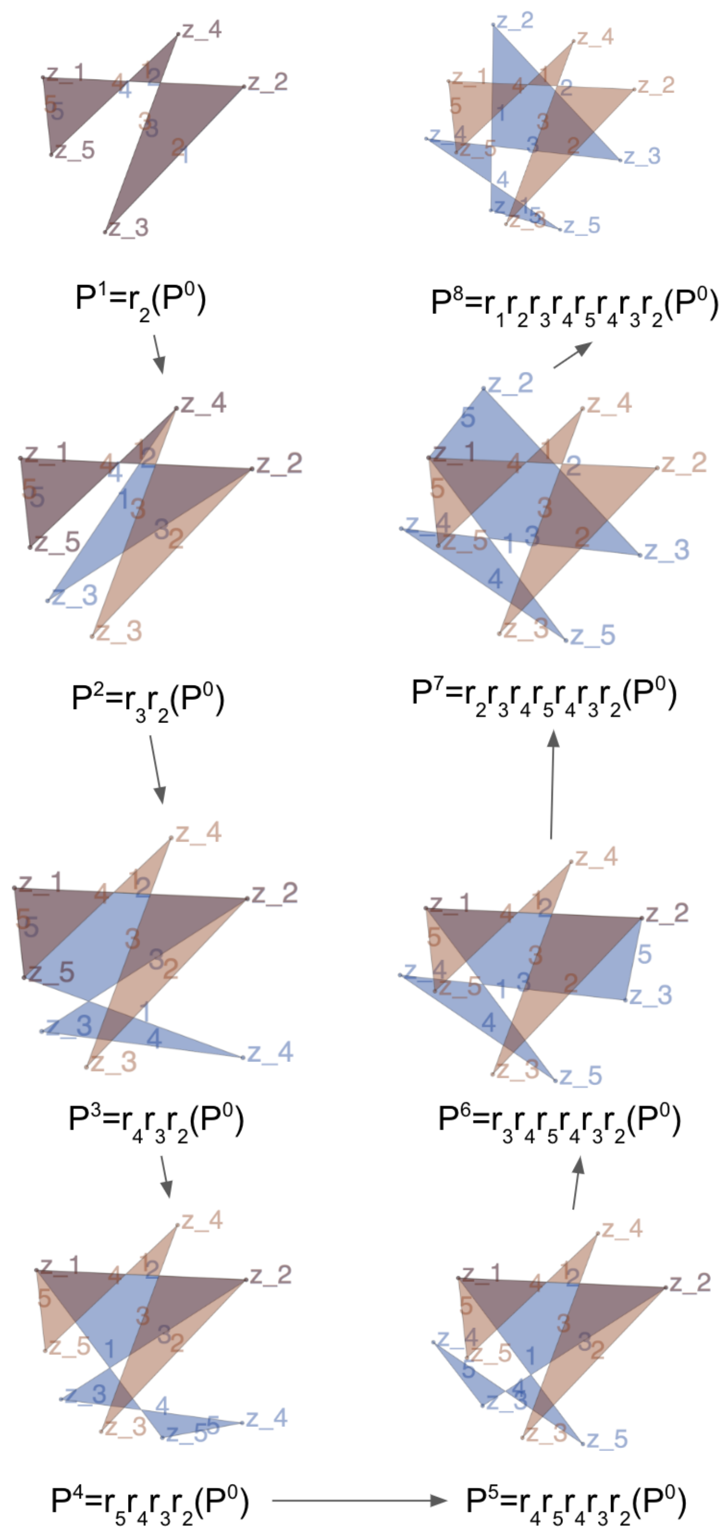

Figure 11: A series of reflection operations performed on the counter-example polygon, where the length of side S_i^0 is labeled as i.

8 Appendix

We used this simulation to explore the transformation r_i studied throughout the paper.

9 Reflection

Throughout the course of this project, one aspect where we struggled the most was making our notation formal and clear. We originally were pretty vague about the definition of a permutation, especially since the length of the sides between any two vertices changed after each reflection operation. By working with the Feng and Susan, we developed more formal notation and rewrote our paper so that it was more formal and clear.

We also received lots of feedback for making our paper more concise. To do so, from draft to draft, we looked at each section and rewrote them, and we also met with the mentors one on one to get more specific feedback on how to be more concise.

10 Who did what

Section 1 (Introduction) - Peter

Section 2 (Assumptions) - Peter

Section 3.1 (Invariance of assumptions) - Written and formalized by Peter, although discussed generally as a group.

Section 3.2 (Invariance of polygon properties) - Written and formalized by Peter, although discussed generally as a group.

Section 4 (Congruence for inscribed and ordered $\widetilde{\mathcal{P}}^0$) - Written by Peter, formalized by Peter and Oscar, although mathematics were discussed as a group.

Section 5 (Algebraic representation) - Peter

Section 6 (A special case: Quadrilateral congruence) - Oscar

Section 7 (Counterexample of the general case) - Cathy