Expander Graphs and their Construction

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Abstract

We study expander graphs, specifically their construction and their applications in network theory. We begin by presenting a method to generate expander graphs via a random d-regular bipartite multigraph, then verify our construction by proving that there exists a 10-regular bipartite multigraph with an expansion of at least 6. Using this construction, we provide a proof of the unique neighbor expansion. We conclude by discussing an application of expander graphs in network theory.

Keywords: Expander Graphs, Network Theory, Robust Networks, Random d-regular Bipartite Multigraphs, Unique Neighbor Expansion

1 Introduction

Expander graphs are graphs whose vertices have constant degrees but that nevertheless exhibit well-connected properties. While it is not immediately obvious that they exist in the first place, we prove they do using a random d-regular bipartite multigraph. We then use this existence to prove other important properties, namely the unique neighbor expansion and an upper bound on the number of rounds needed in order for a message which propagates from vertex to vertex to reach a certain number of vertices. These properties, as alluded to below, are of interest in part due to the physical networks modeled by expander graphs.

In the remainder of this introduction, we first motivate the study of expander graphs by offering a brief overview of their applications. We then provide a brief overview of the wider study of expander graphs in the hopes of conveying how our work fits into the wider body of established knowledge.

In terms of applications, expander graphs are highly related to the construction of networks, such as roads, telephone lines, and the internet. In these instances, there is clearly a benefit to decreasing the number of connections to decrease cost. However, the network must still be well-connected in the sense that cars/phone calls/information can reach their destination efficiently and without risk of a single linkage failure downing the entire network. Therefore, it is important to understand expander graphs, especially their constructions and properties, in order to ensure efficiency in the physical networks they inspire. In addition, expander graphs have numerous applications in computer science, such as in the design of algorithms and error correcting codes [2].

We next describe several high-level trends that exist within previous attempts to study expander graphs. While this information is not needed for the understanding of our subsequent proofs, the curious reader may find interest in this background.

As described in a 2006 survey conducted by Hoory, Linial, and Wigderson, there are four distinct aspects of expander graphs that have been studied over time: extremal problems, typical behavior, explicit constructions, and algorithms. According to the group, "extremal problems focus on the bounding of expansion parameters, while typical behavior problems characterize how the expansion parameters are distributed over random graphs." Finally, "explicit constructions focus on constructing graphs that optimize certain parameters and algorithmic questions study the evaluation and estimation of parameters" [5]. In our paper we turn our attention to a randomized construction and the typical behavior of expander graphs. We simplify matters by restricting ourselves to random d-regular bipartite multigraphs. This use of randomness is a common technique in the study of expander graphs and appears often in proofs related to showing that a given class of graphs is an expander with non-zero probability.

The remainder of this work is divided as follows. Section 2 defines graph-theoretic notation that will serve as a foundation for the sections to follow. Next, Section 3 outlines a construction of a random d-regular multigraph that will be shown to be an expander graph for the case where $d = 10$ in Section 4. Section 5 then provides a proof of the unique neighbor expansion for an expander graph. Lastly, Section 6 uses an expander graph to prove a question related to network theory, namely that a message originating at a vertex and which propagates from a vertex to its neighbors in discrete time steps reaches $\Omega(n)$ vertices in at most $O(\log n)$ rounds.

2 Graph notation

We first define notation that will facilitate the more technical discussion that follows. The purpose of this is twofold: firstly to make our work more accessible to those unfamiliar with graph theory, and secondly to eliminate any confusion with conflicting notation encountered in elsewhere for those who are familiar with graph theory. As will become clear in Section 3, a random d-regular bipartite multigraph will serve as the foundation of our construction of an expander graph.

Definition 1 (Bipartite)**.** *A graph $G = (V, E)$ is said to be bipartite if V can be partitioned into two disjoint sets A, B such that each edge of E has one endpoint in A and one endpoint in B.*

Definition 2 (d-regular)**.** *A graph $G = (V, E)$ is said to be d-regular if for every vertex $v \in V$ it is true that $\deg(v) = d$*

Definition 3 (Multigraph)**.** *A multigraph is a graph which is permitted to have multiple edges.*

We combine the above attributes in the following section to produce a graph that we later prove in Section 4 to be an expander with non-zero probability.

3 Construction of an expander graph

In this section, we first describe the construction of a random graph. We then verify that our construction produces a random d-regular bipartite multigraph.

3.1 Random graph construction

The following construction produces a random graph G on $2n$ vertices via a random assignment of edges. As noted below, Figure 1 above provides an example of these steps.

Step 1: define two vertex sets A, B such that $|A| = |B| = n$. Label these vertices as $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, respectively.

Step 2: define another graph G' as follows. For each vertex $a_i \in A$, create d vertices, $a_{i,1}, \ldots, a_{i,d}$. Do the same for each vertex $b_j \in B$ to produce $b_{j,1}, \ldots, b_{j,d}$. Then define two vertex sets A' (resp. B') to exist on $a_{i,j}$ (resp. $b_{i,j}$) for $i = 1, \ldots, n$ and $j = 1, \ldots, d$. Therefore, $|A'| = |B'| = nd$. Intuitively speaking, think of each of these vertices in G' as corresponding to their "parent" vertex in G.

Step 3: assign edges among the vertices of G' as follows. Create a random permutation $\pi : A' \to B'$ and add an edge between $a_{i,p}$ and $b_{j,q}$ if $\pi(a_{i,p}) = b_{j,q}$.

Step 4: we use G' to define the edges of G. For every $a_i \in A$ and $b_j \in B$, add exactly one edge for every $a_{i,p} \in A'$ that is adjacent to a $b_{j,q} \in B'$. This concludes the construction.

See Figure 1 above for a depiction of these four steps in the case where $n = 3$ and $d = 2$.

3.2 Proof of desired properties

We show that G is d-regular and bipartite. Please note that any use of the term "the construction" refers to the process which we followed to generate G in Section 3.1 above.

We begin by showing that G is d-regular by reasoning about the corresponding properties of G'.

Fact 1. *Graph G is d-regular.*

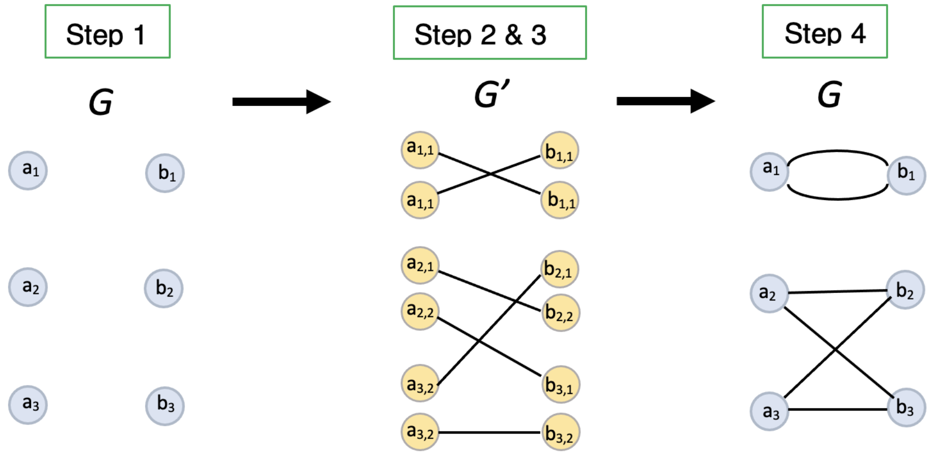

Figure 1: Construction of a random bipartite 2-regular graph on 6 vertices.

Proof. Without loss of generality, consider a vertex $a_i \in A$. We proceed by showing that a_i has a degree of exactly d.

First, recall that in our construction of G' we defined d vertices for every vertex in G, and thought of them as belonging to their parent vertex a_i or b_i in G. After repeating this process for every $a_i \in A$ and $b_i \in B$, we defined a permutation $\pi : A' \to B'$.

Notice that since π is a permutation, it is a bijection. It then follows that every $a_{i,p} \in A'$ maps to *exactly* one $b_{j,q} \in B'$. In other words the d vertices $a_{i,1}, \ldots, a_{i,d}$ will be adjacent to exactly d vertices $b_{j,q} \in B'$ for $q \in \{1, \ldots, d\}$ and $j \in \{1, \ldots, n\}$. We will now uses this one-to-one correspondence to reason about the number of edges of a_i.

Returning to G, recall that we added one edge between a_i and b_j for every $a_{i,p} \in A'$ that is adjacent to some $b_{j,q} \in B'$. By the logic established above, there will be *exactly* d such $b_{j,q} \in B'$. Therefore, a_i will have a degree of d. By symmetry, the same is true for all b_i. Thus every vertex of G will have a degree of d and it follows that G is d-regular from Definition 2. □

We next claim that G is also bipartite.

Fact 2. *Graph G is a bipartite.*

Proof. We proceed to show that G is bipartite on sets A and B.

In our construction of G, we added exactly one edge between $a_i \in A$ and $b_j \in B$ for every $a_{i,p} \in A'$ that is adjacent to some $b_{j,q} \in B'$. Therefore, every edge of G will have one vertex in A and one vertex in B. It then follows that G is bipartite on sets A and B. □

Remark. Note that every graph in the family of random graphs produced from the construction is not necessarily a multigraph. However, we informally refer to G as a multigraph because it it *permitted* to have multiple edges.

4 Expansion of a random graph

In this section, we turn our attention to the expansion of a graph. We first define additional notation related to the expansion of a graph before proving there exists a 10-regular bipartite multigraph with an expansion of at least 6.

4.1 Expansion notation

As first described in Section 1, an expander is a graph with constant degrees but that nevertheless exhibits well-connected properties. We begin by defining a concept related to a graph's expansion.

Definition 4 (Neighborhood)**.** *Given a graph $G = (V, E)$ and a subset of vertices $U \subseteq V$, the neighborhood of vertex set U, denoted $\Gamma(U)$, is the set $\Gamma(U) \subseteq V$ such that every vertex in $\Gamma(U)$ has at least one edge to a vertex in U.*

As will become clear in the definition of a graph's expansion below, our main goal when proving that a graph is an expander is show that every vertex subset U of V has "many" neighbors compared to $|U|$.

Definition 5 (Expansion of a graph)**.** *Given a d-regular graph $G = (V, E)$, its expansion is defined as*

$$\Lambda(G) := \min_{U \subseteq V \ s.t. \ |U| \leq \frac{|V|}{1000}} \frac{|\Gamma(U)|}{|U|} \tag{1}$$

Given a graph's expansion as defined above, we refer to that graph has an expander graph if its expansion is above a certain threshold. In other words, if every not-too-large subset of vertices has many neighbors compared to the size of the subset.

4.2 Proof of expansion

Using our construction described in Section 3.1, we now prove the existence of a specific expander.

Theorem 1. *There exists a random 10-regular multigraph $G = (V, E)$, bipartite on sets A and B where $|A| = |B| = n$ for sufficiently large n, such that $\Lambda(G) \geq 6$.*

Proof. We proceed to show that for a random graph G it holds that $\mathbb{P}[\Lambda(G) < 6] < 1$, thus guaranteeing the existence of an expander. We use two results, namely **Lemma 2** and **Lemma 3**, that are proven in the following section.

For convenience of notation, for a given set U we denote its expansion as $E_U := \frac{|\Gamma(U)|}{|U|}$. Because the expansion of a graph is defined as the minimum of E_U over all subsets U with $|U| \leq \frac{|V|}{1{,}000}$, in order for $\Lambda(G) < 6$ there must exist at least one subset U with $E_U < 6$. We therefore seek $\mathbb{P}[\exists U_A | E_{U_A} < 6]$.

Using **Lemma 2**, proven in Section 4.3 below, it holds for n sufficiently large that

$$\mathbb{P}[\exists U_A | E_{U_A} < 6] < \frac{1}{2}.$$

In other words, **Lemma 2** produces a strict upper bound on the probability that there exists a set $U_A \subset A$ with an expansion less than 6. However, we seek to bound the probability that there exists a set $U \subset V$ with an expansion strictly less than 6, where U can contain vertices in both A and B.

To extend our analysis to the broader case, we use the result of **Lemma 3**, proven in Section 4.3 below, that

$$\mathbb{P}[\exists U | E_U < 6] \leq \mathbb{P}[\exists U_A | E_{U_A} < 6] + \mathbb{P}[\exists U_B | E_{U_B} < 6].$$

We finally produce an upper bound on $\mathbb{P}[\Lambda(G) \geq 6]$. Using the results of **Lemma 2** and **Lemma 3**, we use symmetry on sets A and B to find that

$$\mathbb{P}[\Lambda(G) < 6] \leq \mathbb{P}[\exists U_A | E_{U_A} < 6] + \mathbb{P}[\exists U_B | E_{U_B} < 6] < \frac{1}{2} + \frac{1}{2} < 1.$$

This in turn implies that

$$\mathbb{P}[\Lambda(G) \geq 6] > 0$$

and the proof follows. ∎

4.3 Proof of lemmas used in Theorem 1

Before we seek to bound the probability that there exists a subset U with $|U| \leq \frac{|V|}{1000}$ such that $E_U < 6$, first consider the simpler case of $U \subset A$. More specifically, recall that U is permitted to contain vertices in A and B. Therefore, define sets $U_A = U \cap A$ and $U_B = U \cap B$. We now prove a bound on $\mathbb{P}[E_{U_A} < 6]$ so that we can subsequently produce $\mathbb{P}[\exists U_A | E_{U_A} < 6]$ by union bounding over all such U_A.

Lemma 1. *For a random 10-regular bipartite multigraph and a subset $U_A \subset A$, it holds that*

$$\mathbb{P}[E_{U_A} < 6] \leq \left(\frac{ne}{6u}\right)^{6u} \left(\frac{6u}{n}\right)^{10u}$$

Proof. Given U_A, we first find the probability that all edges from U_A go a given $T \subseteq B$, denoted $\mathbb{P}[U_A \mapsto T]$. For convenience, let $|U_A| = u$ and $|T| = t$.

There are $10u$ edges leaving U_A and $10t$ edges arriving in T, thus there are thus $\binom{10t}{10u}$ possible assignments. Similarly, there are $\binom{10n}{10u}$ total ways to connect the $10u$ edges from U_A to any of the $10n$ edges of B. We then have that

$$\mathbb{P}[U_A \mapsto T] = \frac{\binom{10t}{10u}}{\binom{10n}{10u}}.$$

Using the identity that $\binom{n}{k} \leq \left(\frac{ne}{k}\right)^k$ we can bound $\mathbb{P}[U_A \mapsto T]$ from above by

$$\mathbb{P}[U \mapsto T] = \frac{\binom{10t}{10u}}{\binom{10n}{10u}} \leq \frac{\left(\frac{10te}{10u}\right)^{10u}}{\left(\frac{10ne}{10u}\right)^{10u}} \leq \left(\frac{t}{n}\right)^{10u}. \tag{2}$$

We now seek to bound $\mathbb{P}[E_{U_A} < 6]$ by using the union bound on $\mathbb{P}[U \mapsto T]$ over all T with $u \leq t < 6u$, as this stipulation would guarantee that $E_{U_A} < 6$. Using our previously obtained result for $\mathbb{P}[U \mapsto T]$ in Equation (2), we have that

$$\mathbb{P}[E_{U_A} < 6] \leq \sum_{\substack{T \\ \text{s.t. } t<6u}} \mathbb{P}[U_A \mapsto T] = \sum_{\substack{T \\ \text{s.t. } t<6u}} \left(\frac{t}{n}\right)^{10u}.$$

Over this range of t, notice that any set U that maps to a set T of cardinality less than $6u$ will also map to a set T of cardinality equal to $6u$. We therefore obtain an upper bound by only considering the case where $t = 6u$:

$$\mathbb{P}[E_{U_A} < 6] \leq \sum_{\substack{T \\ \text{s.t. } t=6u}} \left(\frac{6u}{n}\right)^{10u}.$$

As $|B| = n$, there are $\binom{n}{6u} \leq \left(\frac{ne}{6u}\right)^{6u}$ ways of choosing this set T of size $6u$. Therefore, we can re-express the summation as

$$\mathbb{P}[E_{U_A} < 6] \leq \left(\frac{ne}{6u}\right)^{6u} \left(\frac{6u}{n}\right)^{10u}$$

and the proof follows. □

Since the above lemma provides an upper bound on $\mathbb{P}[E_{U_A} < 6]$ for given U_A, we now turn our attention to union bounding over all such U_A such that $1 \leq u \leq \frac{|V|}{1000} = \frac{n}{500}$.

Lemma 2. *For n sufficiently large,*

$$\mathbb{P}[\exists U_A | E_{U_A} < 6] < \frac{1}{2}. \tag{3}$$

Proof. We first claim that

$$\mathbb{P}[\exists U_A | E_{U_A} < 6] \leq \sum_{u=1}^{\lfloor n/500 \rfloor} \sum_{\substack{U_A \\ \text{s.t. } |U_A|=u}} \mathbb{P}[E_{U_A} < 6]. \tag{4}$$

Notice that this is precisely the union bound of $\mathbb{P}[E_{U_A} < 6]$ over all sets U_A such that $|U_A| = u$ for $u \leq \frac{n}{500}$. It therefore produces an upper bound on the probability that there exists at least one U_A with $E_{U_A} < 6$, which is what we seek in Equation 3.

We next produce a bound for the inner summation in Equation 4 above. Since $|A| = n$, there are $\binom{n}{u} \le \left(\frac{ne}{u}\right)^u$ subsets U_A such that $|U_A| = u$. From this and the result obtained in Lemma 1, we combine like terms to find that

$$\sum_{\substack{U_A \\ \text{s.t. } |U_A|=u}} \mathbb{P}[E_{U_A} < 6] \le \left(\frac{ne}{u}\right)^u \left(\frac{ne}{6u}\right)^{6u} \left(\frac{6u}{n}\right)^{10u} \le \left(\frac{6^4 e^7 u^3}{n^3}\right)^u$$

With n sufficiently large, we notice that this upper bound approaches 0 as u becomes large. We verify this is the case by arbitrarily setting $n = 10000$, taking the derivative with respect to u, and noting that the expression archives a maximum at $u = 1$. Therefore for n sufficiently large we can set $u = 1$ to obtain

$$\sum_{\substack{U_A \\ \text{s.t. } |U_A|=u}} \mathbb{P}[E_{U_A} < 6] \le \left(\frac{6^4 e^7 u^3}{n^3}\right)^u \le \frac{6^4 e^7}{n^3}.$$

Plugging this result in the outer summation of Equation 4, we have

$$\mathbb{P}[\exists U_A | E_{U_A} < 6] \le \sum_{u=1}^{\lfloor n/500 \rfloor} \frac{6^4 e^7}{n^3}.$$

We now rewrite the summation to obtain a strict upper bound. First note that because $\lfloor \frac{n}{500} \rfloor < \lfloor \frac{n}{499} \rfloor$ for $n > 0$, we can write

$$\sum_{u=1}^{\lfloor n/500 \rfloor} \frac{6^4 e^7}{n^3} < \sum_{u=1}^{\lfloor n/499 \rfloor} \frac{6^4 e^7}{n^3}.$$

Noting that the expression inside the summation does not depend on u and will run less than or equal to $\frac{n}{499}$ times, we have that

$$\sum_{u=1}^{\lfloor n/499 \rfloor} \frac{6^4 e^7}{n^3} \le \frac{n}{499} \cdot \frac{6^4 e^7}{n^3}.$$

For $n \ge \sqrt{\frac{2 \cdot 6^4 e^7}{499}}$, then

$$\frac{n}{499} \cdot \frac{6^4 e^7}{n^3} \le \frac{1}{2}.$$

We have thus shown for n sufficiently large that

$$\mathbb{P}[\exists U_A | E_{U_A} < 6] < \frac{1}{2}.$$

□

Lemma 3. *Given a set $U = U_A \cup U_B$ it holds that*

$$\mathbb{P}[\exists U | E_U < 6] \le \mathbb{P}[\exists U_A | E_{U_A} < 6] + \mathbb{P}[\exists U_B | E_{U_B} < 6].$$

Proof. From G being bipartite, we know the neighborhoods of a vertex in A and a vertex in B will be disjoint, thus $|\Gamma(U)| = |\Gamma(U_A)| + |\Gamma(U_B)|$. Using the fact that $|U| = |U_A| + |U_B|$, we then write that

$$\begin{aligned} E_U = \frac{|\Gamma(U)|}{|U|} = \frac{|\Gamma(U_A)| + |\Gamma(U_B)|}{|U_A| + |U_B|} &= \frac{|\Gamma(U_A)|}{|U_A| + |U_B|} + \frac{|\Gamma(U_B)|}{|U_A| + |U_B|} \\ &\le \frac{|\Gamma(U_A)|}{|U_A|} + \frac{|\Gamma(U_B)|}{|U_B|} = E_{U_A} + E_{U_B} \end{aligned}$$

taking the probability of both sides gives

$$\mathbb{P}[E_U < 6] \le \mathbb{P}[E_{U_A} + E_{U_B} < 6]$$

which can be rewritten using the union bound to obtain

$$\mathbb{P}[E_U < 6] \le \mathbb{P}[E_{U_A} \le 6] + \mathbb{P}[E_{U_B} \le 6].$$

If we proceed to union bound over all such subsets, U, U_A, U_B in a similar approach used in Equation 4 of **Lemma 2**, we have that

$$\sum_{u=1}^{\lfloor n/500 \rfloor} \sum_{\substack{U \\ \text{s.t. } |U_A|=u}} \Big(\mathbb{P}[E_U < 6\Big) \le \sum_{u=1}^{\lfloor n/500 \rfloor} \sum_{\substack{U_A, U_B \\ \text{s.t. } |U_A|=|U_B|=u}} \Big(\mathbb{P}[E_{U_A} \le 6] + \mathbb{P}[E_{U_B} \le 6]\Big)$$

which implies

$$\mathbb{P}[\exists U | E_U < 6] \le \mathbb{P}[\exists U_A | E_{U_A} < 6] + \mathbb{P}[\exists U_B | E_{U_B} < 6].$$

□

5 Unique neighbor expansion

Using our construction in Section 3.1 and our proof of expansion in Section 4.2, we now provide a proof of the unique neighbor expansion property. As described below, this will guarantee that there will be "many" nodes that are connected by an edge to one and only one node in U.

Theorem 2 (Unique Neighbor Expander)**.** *Given a random d-regular multigraph $G = (V, E)$ which is bipartite on vertex sets A and B with an expansion of λ, then for some constant α with $2\lambda - \alpha \ge d$ it holds for every subset $U \subset A$ that $|\Gamma_u(U)| \ge \alpha|U|$ where $\Gamma_u(U)$ contains the neighbors of U that have exactly one edge connecting them to a node in U.*

Proof. We proceed to show that if graph G is an expander, then it must also be a unique neighbor expander. More specifically, if there are too many vertices in that are not unique neighbors of U then the condition that G is d−regular will be violated.

First, let $|U| = u$, and let $\Gamma_n(U) \subset B$ contain the nodes in B with at least two edges connecting them to a node in U. Noting that $\Gamma_u(U)$ and $\Gamma_n(U)$ are disjoint and $\Gamma(U) = \Gamma_u(U) \cup \Gamma_u(U)$,

$$|\Gamma(U)| = |\Gamma_u(U)| + |\Gamma_u(U)|. \tag{5}$$

Next, let $\Phi(U, T)$ denote the number of edges between the number of edges between some $U \subset A$ and some set $T \subseteq B$. From Equation 5 we obtain

$$\Phi\Big(U, \Gamma(U)\Big) = \Phi\Big(U, \Gamma_u(U)\Big) + \Phi\Big(U, \Gamma_n(U)\Big). \tag{6}$$

Since every vertex in $\Gamma_u(U)$ is connected to exactly 1 vertex in U, it follows that $\Phi(U, \Gamma_u(U)) = |\Gamma_u(U)|$. In a similar approach, from the fact that every vertex in $\Gamma_n(U)$ is connected to at least 2 vertices in U, it follows that $\Phi(U, \Gamma_n(U) \ge 2|\Gamma_n(U)|$. Plugging these two results into Equation 6,

$$\Phi\Big(U, \Gamma(U)\Big) \ge |\Gamma_u(U)| + 2|\Gamma_n(U)|.$$

From d-regularity it must hold that $\Phi\Big(\Gamma(U), T\Big) = ud$, therefore

$$ud \ge |\Gamma_u(U)| + 2|\Gamma_n(U)|. \tag{7}$$

Towards a contradiction, suppose for some set U and constant $c > 0$ that

$$|\Gamma_u(U)| = \alpha u - c. \tag{8}$$

By the fact that graph G is an expander, we know that $|\Gamma(U)| \ge \lambda u$. Plugging this result and our assumption into Equation 5,

$$\lambda u = (\alpha u - c) + |\Gamma_n(U)|$$

so

$$|\Gamma_n(U)| = u(\lambda - \alpha) + c. \tag{9}$$

Plugging in our results obtained in Equation 8 and 9 into Equation 7 this above expression gives

$$ud \ge (\alpha u - c) + 2(\lambda u - \alpha u + c)$$

so

$$ud \ge u(2\lambda - \alpha) + c.$$

Using the fact that $2\lambda - \alpha \ge d$ we obtain

$$ud \ge ud + c \text{ for some } c > 0$$

and the proof follows by way of contradiction. □

6 Transmission among vertices of an expander graph

As previously mentioned in Section 1, expander graphs often arise in network theory. For example, consider the scenario in which a node of a graph wishes to broadcast a message to other nodes, and thus sends the message to all its neighbors, who in turn broadcast the message in the next round to all their neighbors, and so on. Among other things, one might wonder how quickly the message reaches at least a certain number of nodes. To answer this question we present the following theorem.

Theorem 3. *Suppose $G = (V, E)$ is a random, d-regular bipartite multigraph as described in Section 3 with an expansion of λ. Then if a node s broadcasts a messages to its neighbors and so on, then the message reaches $\Omega(n)$ nodes in at most $O(\log n)$ rounds.*

Proof. We wish to show that there exists constants c_1 and c_2 such that message reaches at least $c_1 n$ nodes in at most $c_2 \log n$ rounds. To accomplish this, we leverage the the expansion properties of G.

For a given step i, define the set M_i to contain all nodes that have the message. If we assume that $c_1 = \frac{1}{500}$, then we are restricted to the case where $M_i \leq \frac{|V|}{1000}$ and can thus employ the definition of expansion in Equation 1 to write that

$$\frac{|\Gamma(M_i)|}{|M_i|} \geq \lambda$$

so

$$|\Gamma(M_i)| \geq \lambda |M_i|.$$

Since $\Gamma(M_i)$ contains the vertices with at least one edge to a node in M_i, all nodes in $\Gamma(M_i)$ will have the message in round $i + 1$. Therefore

$$M_{i+1} \geq \lambda M_i$$

If we assume that a single node s begins with the message at the 0^{th} step, so that $|M_0| = 1$, then in step i

$$|M_i| \geq \lambda^i.$$

Setting $|M_i| = c_1 n$ and solving for i gives

$$c_1 n \geq \lambda^i$$

so

$$c_2 \log_\lambda n \geq i.$$

Therefore the message reaches at least $c_1 n$ nodes in at most $c_2 \log n$ rounds. Or equivalently, the message reaches $\Omega(n)$ nodes in at most $O(n)$ round. □

7 Extension: pool testing

While not discussed in the main section of our paper above, we would like to note a third important application of expander graphs related to virus testing methodologies.

Suppose there is a set A of n people, each of which may have a certain disease. In order to detect which individuals from A have the virus, we take random groups of d people from A, pool their biological samples, and run the pooled test. We repeat this process n times in such a way that every individual is tested exactly d times. However, each pooled test only returns positive if *exactly* one person has the disease. If we are given some upper bound on the number of people in A that have the virus, we might wonder if our method succeeds at detecting the virus.

While we do not provide a formal proof of such success, we speculate the affirmative. First note that the graph described by the above scenario is d-regular and bipartite on sets A and B each of cardinality n. Therefore, our results from Section 4.2 and Section 5 apply.

One possible approach is as follows. For a vertex $w \in B$, we say that w "fails" if at least 2 of the d vertices in A that w is adjacent to have the virus. If we then suppose exactly m people in A have the disease, then for a fixed m we conjecture that the probability that w fails goes to 0. We then surmise that there may exist a backtracking method to deduce which vertices in A have the virus, however we leave this task up to future work.

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